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All unit-regular elements of relational hypersubstitutions for algebraic systems*

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Abstract. Relational hypersubstitutions for algebraic systems are mappings which map operation symbols to terms and map relation symbols to relational terms preserving arities. The set of all relational hypersubstitutions for algebraic systems together with a binary operation defined on this set forms a monoid. In this paper, we determine all unit-regular elements on this monoid of type $((m),(n))$ for arbitrary natural numbers $m, n \geq 2$.

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1. Introduction

The notation of a hypersubstitution of type τ for universal algebras was introduced by K. Denecke et al. [2]. To recall the concept of a hypersubstitution of type τ, we recall first the concept of an m-ary term of type τ. Let $(f_i)_{i\in I}$ be a sequence of operation symbols indexed by the set I. For every operation symbol f_i, we assign a natural number $n_i \in \mathbb{N}^+ := \mathbb{N}\setminus\{0\}$, called the arity of f_i. The sequence $\tau = (n_i)_{i\in I}$ of arities of f_i is called the type.

Let $X := \{x_1, \ldots, x_n, \ldots\}$ be a countably infinite set of symbols called

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variables. For each $n \geq 1$, let $X_m := \{x_1, \ldots, x_m\}$. An *m-ary term of type* τ is defined inductively as follows.

(i) Every variable $x_k \in X_m$ is an m-ary term of type τ.

(ii) If $t_1, \ldots, t_{n_i}$ are m-ary terms of type τ and f_i is an n_i-ary operation symbol, then $f_i(t_1, \ldots, t_{n_i})$ is an m-ary term of type τ.

Let $W_\tau(X_m)$ be the set of all m-ary terms of type τ which contains $x_1, \ldots, x_m$ and is closed under finite application of (ii) and let $W_\tau(X) := \bigcup_{m \in \mathbb{N}^+} W_\tau(X_m)$ be the set of all terms of type τ.

A hypersubstitution of type τ is a mapping $\sigma : \{f_i | i \in I\} \to W_\tau(X)$ preserving the arity. Let $Hyp(\tau)$ be the set of all hypersubstitutions of type τ. To define a binary operation on this set, we define inductively the concept of a superposition of terms $S_n^m : W_\tau(X_m) \times (W_\tau(X_n))^m \to W_\tau(X_n)$ as follows.

(i) If $t = x_k$ for $1 \leq k \leq m$, then $S_n^m(x_k, s_1, \ldots, s_m) := s_k$.

(ii) If $t = f_i(t_1, \ldots, t_{n_i})$, then
$S_n^m(t, s_1, \ldots, s_m) := f_i(S_n^m(t_1, s_1, \ldots, s_m), \ldots, S_n^m(t_{n_i}, s_1, \ldots, s_m))$.

For every $\sigma \in Hyp(\tau)$, we define a mapping $\widehat{\sigma} : W_\tau(X) \to W_\tau(X)$ as follows:

(i) $\widehat{\sigma}[x] := x \in X$,

(ii) $\widehat{\sigma}[f_i(t_1 \ldots, t_{n_i})] := S_m^{n_i}(\sigma(f_i), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}])$, for any n_i-ary operation symbol f_i and $\widehat{\sigma}[t_j]$ are already defined for all $1 \leq j \leq n_i$.

Further, a binary operation $\circ_h$ on $Hyp(\tau)$ is defined by $\sigma \circ_h \alpha = \hat{\sigma} \circ \alpha$, where $\circ$ denotes the usual composition of mappings. Then one can prove that $(Hyp(\tau), \circ_h, \sigma_{id})$ is a monoid, where $\sigma_{id}(f_i) = f_i(x_1, x_2, ..., x_{n_i})$ is the identity element.

On the other hand, we can consider algebraic systems in the sense of Mal'cev [5]. An algebraic system of type $(\tau, \tau^{'})$ is a triple

$$(A, (f_i^A)_{i\in I}, (\gamma_j^A)_{j\in J})$$

consisting of a nonempty set A, a sequence $(f_i^A)_{i\in I}$ of n_i-ary operations defined on A and a sequence $(\gamma_j^A)_{j\in J}$ of n_j-ary relations on A, where $\tau = (n_i)_{i\in I}$ is a sequence of the arity of each operation f_i^A and $\tau^{'} = (n_j)_{j\in J}$ is a sequence of the arity of each relation γ_j^A. The pair $(\tau, \tau^{'})$ is called the type of an algebraic system, see more detail in [6], [7].

In 2008 [3], K. Denecke and D. Phusanga introduced the concept of a hypersubstitution for algebraic systems which is a mapping that assigns an operation symbol to a term and assigns a relation symbol to a formula which preserve the arity. The set of all hypersubstitutions for algebraic systems of type $(\tau, \tau^{'})$ is denoted by $Hyp(\tau, \tau^{'})$. They defined an assosiative operation $\circ_r$ on this set and proved that $(Hyp(\tau, \tau^{'}), \circ_r, \sigma_{id})$ forms a monoid where σ_{id} is an identity hypersubstitution for algebraic systems.

In 2018, D. Phusanga and J. Koppitz [8] introduced the concept of a relational hypersubstitution for algebraic systems of type (τ, τ') and proved that this set together with an assosiative binary operation and the identity element forms a monoid. The aim of the present paper is to determine the set of all unit-regular elements of relational hypersubstitutions of type $((m), (n))$ for arbitrary natural numbers $m, n \geq 2$.

Let (τ, τ') be a type. An n-ary relational term of type (τ, τ') and a relational hypersubstitution for algebraic systems are defined as follows.

Definition 1 [6]. Let J be an indexed set. If $j \in J$ and $t_1, t_2, ..., t_{n_j}$ are $n-$ary terms of type τ and γ_j is an n_j-ary relation symbol, then $\gamma_j(t_1, t_2, ..., t_{n_j})$ is an $n-$ary relational term of type $(\tau, \tau^{'})$.

Let $\gamma F_{(\tau,\tau')}(X_n)$ be the set of all $n-$ary relational terms of type $(\tau,\tau^{'})$ and let $\gamma F_{(\tau,\tau')}(X) := \cup_{n\in\mathbb{N}}\gamma F_{(\tau,\tau')}(X_n)$ be the set of all relational terms of type $(\tau,\tau^{'})$.

A relational hypersubstitution for algebraic systems of type $(\tau,\tau^{'})$ is a mapping

$$\sigma : \{f_i|\ i\in I\}\cup\{\gamma_j|\ j\in J\}\to W_\tau(X)\cup\gamma F(\tau,\tau^{'})(X)$$

with $\sigma(f_i)\in W_\tau(X_{n_i})$ and $\sigma(\gamma_j)\in\gamma F_{(\tau,\tau')}(X_{n_j})$. The set of all relational hypersubstitutions for algebraic systems of type $(\tau,\tau^{'})$ is denoted by $Relhyp(\tau,\tau^{'})$.

To define a binary operation on this set, we give the concept of a superposition of relational terms. A superposition of relational terms $R^m_n : (W_\tau(X)\cup\gamma F_{(\tau,\tau')}(X_m))\times(W_\tau(X_n))^m\to W_\tau(X)\cup\gamma F_{(\tau,\tau')}(X_m)$ is defined as follows, for $t,t_1,\ldots,t_{m_i}\in W_\tau(X_m)$, $s_1,\ldots,s_m\in W_\tau(X_n)$,

(i) $R^m_n(t,s_1,\ldots,s_m) := S^m_n(t,s_1,\ldots,s_m)$,

(ii) $R^m_n(F,s_1,\ldots,s_m) := \gamma_j(S^m_n(t_1,s_1,\ldots,s_m),\ldots,S^m_n(t_{n_j},s_1,\ldots,s_m))$.

Every relational hypersubstitution for algebraic systems σ can be extended to a mapping $\widehat{\sigma} : W_\tau(X)\cup\gamma F_{(\tau,\tau')}(X)\to W_\tau(X)\cup\gamma F_{(\tau,\tau')}(X)$ as follows.

(i) $\widehat{\sigma}[x_i] := x_i$ for $i\in\mathbb{N}$,

(ii) $\widehat{\sigma}[f_i(t_1\ldots,t_{n_i})] := S^{n_i}_m(\sigma(f_i),\widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_{n_i}])$, where $i\in I$ and $t_1,\ldots,t_{n_i}\in W_\tau(X_m)$, i.e., any occurrence of the variable x_k in $\sigma(f_i)$ is replaced by the term $\widehat{\sigma}[t_k], 1\leq k\leq n_i$,

(iii) $\widehat{\sigma}[\gamma_j(s_1\ldots,s_{n_j})] := R^{n_j}_n(\sigma(\gamma_j),\widehat{\sigma}[s_1],\ldots,\widehat{\sigma}[s_{n_j}])$, where $j\in J$ and $s_1,\ldots,s_{n_j}\in W_\tau(X_n)$, i.e., any occurrence of the variable x_k in $\sigma(\gamma_j)$ is replaced by the term $\widehat{\sigma}[s_k], 1\leq k\leq n_j$.

We define a binary operation $\circ_r$ on $Relhyp(\tau, \tau^{'})$ by $\sigma \circ_r \alpha := \hat{\sigma} \circ \alpha$ where $\circ$ is the usual composition of mappings and $\sigma, \alpha \in Relhyp(\tau, \tau^{'})$.

Let σ_{id} be the relational hypersubstitution which maps each m_i-ary operation symbol f_i to the term $f_i(x_1, \ldots, x_{m_i})$ and maps each n_j-ary relation symbol γ_j to the relational term $\gamma_j(x_1, \ldots, x_{n_j})$. D. Phusanga and J. Koppitz [3] proved that $(Relhyp(\tau, \tau^{'}), \circ_h, \sigma_{id})$ is a monoid.

Definition 2. Let $(\tau, \tau') = ((m), (n))$ be a type with an m-ary operation symbol f and an n-ary relation symbol γ. Let $\sigma_{t,F}$ be the relational hypersubstitution σ of type $((m), (n))$ with maps f to the term $t \in W_{(m)}(X_m)$ and maps γ to the relational term $F \in \gamma F_{((m),(n))}(X_n)$.

A relational hypersubstitution for algebraic systems of type $((m), (n))$ is called *a projection relational hypersubstitution for algebraic systems of type* $((m), (n))$ if the term $\sigma_{t,F}(f)$ is a variable.

Let $ProjR((m), (n))$ be the set of all projection relational hypersubstitutions for algebraic systems of type $((m), (n))$.

A relational hypersubstitution for algebraic systems of type $((m), (n))$ is called *a pre-relational hypersubstitution for algebraic systems of type* $((m), (n))$ if the term $\sigma_{t,F}(f)$ is not a variable.

Let $PreR((m), (n))$ be the set of all pre-relational hypersubstitutions for algebraic systems of type $((m), (n))$.

Proposition 1 [1]. *The set* $ProjR((m), (n)) \cup \{\sigma_{id}\}$, $PreR((m), (n))$ *are submonoids of* $Relhyp((m), (n))$.

In 2015, W. Wongpinit and S. Leeratanavalee [9] introduced the concept of the $i - most$ of terms as follows.

Definition 3 [9]. Let $\tau = (m)$ be a type with an m-ary operation symbol f, $t \in W_{(m)}(X)$ and $1 \leq i \leq m$. An $i - most(t)$ is defined inductively as follows:

(i) If t is a variable, then $i-most(t)=t$.

(ii) If $t=f(t_1,\ldots,t_n)$ where $t_1,\ldots,t_n\in W_{(m)}(X)$, then $i-most(t):=i-most(t_i)$.

Example 1. Let $\tau=(3)$ be a type, $t=f(x_3,f(x_2,x_1,x_3),f(x_3,x_1,x_2))$. Then $1-most(t)=x_3$, $2-most(t)=2-most(f(x_2,x_1,x_3))=x_1$ and $3-most(t)=3-most(f(x_3,x_1,x_2))=x_2$.

Lemma 1 [9]. *Let $s,t\in W_{(m)}(X)$. If $j-most(t)=x_k\in X_m$ and $k-most(s)=x_i$, then $j-most(\hat{\sigma}_t[s])=x_i$.*

The above lemma can be applied to any relational hypersubstitution for algebraic systems of type $((m),(n))$.

Let $\sigma_{t,F}$ be a relational hypersubstitution for algebraic systems of type $((m),(n))$, where $t\in W_{(m)}(X_m)$ and $F\in\gamma F_{((m),(n))}(X_n)$, $s\in W_{(m)}(X_m)$. We have if $i-most(t)=x_j$, then $i-most(\widehat{\sigma}_{t,F}[s])=j-most(s)$.

2. Unit-regular elements in $Relhyp((m),(n))$

Let $(\tau,\tau^{'})=((m),(n))$ be a type with an $m-$ary operation symbol f, an $n-$ary relation symbol γ, $t\in W_{(m)}(X_m)$ and $F\in\gamma F_{((m),(n))}(X_n)$, we denote

$var(t)$:= the set of all variables occurring in the term t.

$var(F)$:= the set of all variables occurring in the relational term F.

Let $\sigma_{t,F}\in Relhyp((m),(n))$, we denote

$R_X:=\{\sigma_{t,F}\ |t=x_i$ and $F=\gamma(s_1,...,s_n)$ with $var(F)=\{x_{b_1},...,x_{b_l}\}$ such that $j-most(s_{b_k^{'}})=x_{b_k}$ for all $k=1,...,l$ and some distinct $b_1^{'},...,b_l^{'}\in\{1,...,n\}$ where $j\in\{1,...,m\}\}$;

$R_T := \{\sigma_{t,F} \mid t = f(t_1, \ldots, t_m)$ and $F = \gamma(s_1, ..., s_n)$ with $var(t) = \{x_{a_1}, ..., x_{a_k}\}$ and $var(F) = \{x_{b_1}, ..., x_{b_l}\}$ such that $t_{a'_i} = x_{a_i}$ and $s_{b'_j} = x_{b_j}$ for all $i = 1, ..., k, j = 1, ..., l$ for some distinct $a'_1, ..., a'_k \in \{1, ..., m\}$ and for some distinct $b'_1, ..., b'_l \in \{1, ..., n\}\}$.

In 2019, J. Daengsaen and S. Leeratanavalee [6] characterized the regular relational hypersubstitutions for algebraic systems of type $(\tau, \tau') = ((m), (n))$.

Lemma 2. *Let* $t = x_i \in X_m$, $F = \gamma(s_1, ..., s_n) \in \gamma\mathcal{F}_{((m),(n))}(X_n)$ *and* $\sigma_{t,F} \in ProjR((m), (n))$. *Then* $\sigma_{t,F}$ *is not unit.*

Proof. Assume $t = x_i \in X_m$ and $F = \gamma(s_1, ..., s_n) \in \gamma\mathcal{F}_{((m),(n))}(x_n)$. Let $\sigma_{u,H} \in Relhyp((m), (n))$.

Case I. $\sigma_{u,H} \in ProjR((m), (n))$, i.e., $u = x_j \in X_m$ and $H = \gamma(h_1, ..., h_n) \in \gamma\mathcal{F}_{((m),(n))}(X_n)$. Then, $(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{t,F}[u] = \widehat{\sigma}_{t,F}[x_j] = x_j$.

Case II. $\sigma_{u,H} \in PreR((m), (n))$, i.e., $u = f(u_1, ..., u_m) \in W_{(m)}(X_m)$ and $H = \gamma(h_1, ..., h_n) \in \gamma\mathcal{F}_{((m),(n))}(X_n)$.

Consider

$$
\begin{aligned}
(\sigma_{t,F} \circ_r \sigma_{u,H})(f) &= \widehat{\sigma}_{t,F}[f(u_1, ..., u_m)] \\
&= S^m_m(t, \widehat{\sigma}_{t,F}[u_1], ..., \widehat{\sigma}_{t,F}[u_m]) \\
&= S^m_m(x_i, \widehat{\sigma}_{t,F}[u_1], ..., \widehat{\sigma}_{t,F}[u_m]) \\
&= \widehat{\sigma}_{t,F}[u_i] \\
&= x_l \in X_m.
\end{aligned}
$$

Thus $\sigma_{t,F} \circ_r \sigma_{u,H} \neq \sigma_{id}$ for all $\sigma_{u,H} \in Relhyp((m), (n))$. Therefore $\sigma_{t,F}$ is not unit. □

Lemma 3. *Let* $\sigma_{t,F} \in PreR((m), (n))$. *If* $t_i \in W_{(m)}(X_m)\backslash(X_m)$ *for some* $i \in \{1, ..., m\}$ *and* $s_j \in W_{(m)}(X_n)\backslash(X_n)$ *for some* $j \in \{1, ..., n\}$, *then* $\sigma_{t,F}$

is not unit.

Proof. Let $t_i \in W_{(m)}(X_m)\backslash(X_m)$ for some $i \in \{1, ..., m\}$ and $s_j \in W_{(m)}(X_n)\backslash(X_n)$ for some $j \in \{1, ..., n\}$. Let $\sigma_{u,H} \in Relhyp((m),(n))$ where $u = f(u_1, ..., u_m) \in W_{(m)}(X_m)$ and $H = \gamma(h_1, ..., h_n) \in \gamma\mathcal{F}_{((m),(n))}(X_n)$.

Consider

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(f) &= \widehat{\sigma}_{t,F}[f(u_1, ..., u_m)] \\ &= S^m_m(f(t_1, ..., t_m), \widehat{\sigma}_{t,F}[u_1], ..., \widehat{\sigma}_{t,F}[u_m]) \\ &= f(S^m_m(t_1, \widehat{\sigma}_{t,F}[u_1], ..., \widehat{\sigma}_{t,F}[u_m]), ..., \\ &\quad S^m_m(t_m, \widehat{\sigma}_{t,F}[u_1], ..., \widehat{\sigma}_{t,F}[u_m])).\end{aligned}$$

Since $t_i \in W_{(m)}(X_m)\backslash(X_m)$, so $\widehat{\sigma}_{t,F}[u_j] \in W_{(m)}(X_m)\backslash(X_m)$ for all $j \in \{1, ..., m\}$. Then $(\sigma_{t,F} \circ_r \sigma_{u,H})(f) \neq f(x_1, ..., x_m) = \sigma_{id}(f)$, and

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= \widehat{\sigma}_{t,F}[\gamma(h_1, ..., h_n)] \\ &= R^n_n(\gamma(s_1, ..., s_n), \widehat{\sigma}_{t,F}[h_1], ..., \widehat{\sigma}_{t,F}[h_n]) \\ &= \gamma(S^n_n(s_1, \widehat{\sigma}_{t,F}[h_1], ..., \widehat{\sigma}_{t,F}[h_n]), ..., \\ &\quad S^n_n(s_n, \widehat{\sigma}_{t,F}[h_1], ..., \widehat{\sigma}_{t,F}[h_n])).\end{aligned}$$

Since $t_i \in W_{(m)}(X_m)\backslash(X_m)$, so $\widehat{\sigma}_{t,F}[h_j] \in W_{(m)}(X_m)\backslash(X_m)$ for all $i \in \{1, ..., n\}$. Then $(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) \neq f(x_1, ..., x_n) = \sigma_{id}(\gamma)$.

Hence $\sigma_{t,F} \circ_r \sigma_{u,H} \neq \sigma_{id}$ for all $\sigma_{u,H} \in Relhyp((m),(n))$. Therefore $\sigma_{t,F}$ is not unit. □

Theorem 1. *Let $\sigma_{t,F} \in Relhyp((m),(n))$. Then $\sigma_{t,F}$ is unit if and only if $t = f(x_{\pi(1)}, ..., x_{\pi(m)})$ and $F = \gamma(s_{\phi(1)}, ..., s_{\phi(n)})$ where π, ϕ are bijections on $\{1, ..., m\}$ and $\{1, ..., n\}$, respectively.*

Proof. Assume that $\sigma_{t,F}$ is a unit element. Then there exists $\sigma_{u,H} \in Relhyp((m),(n))$ such that $\sigma_{t,F} \circ_r \sigma_{u,H} = \sigma_{id} = \sigma_{u,H} \circ_r \sigma_{t,F}$.

By Lemma 3, we get $\sigma_{t,F}, \sigma_{u,H} \in R_T$. So $t = f(t_1, ..., t_m)$, $u = f(u_1, ..., u_m)$ where $t_1, ..., t_m, u_1, ..., u_m \in \{x_1, ..., x_m\}$ and $F = \gamma(s_1, ..., s_n)$, $H = \gamma(h_1, ..., h_n)$ where $s_1, ..., s_n, h_1, ..., h_n \in \{x_1, ..., x_n\}$.

Let $t = f(x_{\pi(1)}, ..., x_{\pi(m)})$, $F = \gamma(s_{\phi(1)}, ..., s_{\phi(n)})$ and $u = f(x_{\pi'(1)}, ..., x_{\pi'(m)})$, $H = \gamma(x_{\phi'(1)}, ..., x_{\phi'(n)})$ where $\pi, \pi^{'}$ are bijections on $\{1, ..., m\}$ and $\phi, \phi^{'}$ are bijections on $\{1, ..., n\}$.

Consider

$$\begin{aligned}
\sigma_{id} &= (\sigma_{t,F} \circ_r \sigma_{u,H})(f) \\
&= \widehat{\sigma}_{t,F}[f(x_{\pi'(1)}, ..., x_{\pi'(m)})] \\
&= S^m_m(f(x_{\pi(1)}, ..., x_{\pi(m)}), x_{\pi'(1)}, ..., x_{\pi'(m)}) \\
&= f(x_{\pi'(\pi(1))}, ..., x_{\pi'(\pi(m))}) \\
&= f(x_{(\pi'\circ\pi)(1)}, ..., x_{(\pi'\circ\pi)(m)})
\end{aligned}$$

and

$$\begin{aligned}
\sigma_{id} &= (\sigma_{u,H} \circ_r \sigma_{t,F})(f) \\
&= \widehat{\sigma}_{u,H}[f(x_{\pi(1)}, ..., x_{\pi(m)})] \\
&= S^m_m(f(x_{\pi'(1)}, ..., x_{\pi'(m)}), x_{\pi(1)}, ..., x_{\pi(m)}) \\
&= f(x_{\pi(\pi'(1))}, ..., x_{\pi(\pi'(m))}) \\
&= f(x_{(\pi\circ\pi')(1)}, ..., x_{(\pi\circ\pi')(m)}).
\end{aligned}$$

Then $\pi \circ \pi^{'} = (1) = \pi^{'} \circ \pi$ and $\pi \circ \pi^{'}, \pi^{'} \circ \pi$ are bijections.

Next, we will show that π is a bijection. Let $\pi(i) = \pi(j)$ for some $i, j \in \{1, ..., m\}$. Then $(\pi^{'} \circ \pi)(i) = \pi^{'}(\pi(i)) = \pi^{'}(\pi(j)) = (\pi^{'} \circ \pi)(j)$. Since $(\pi^{'} \circ \pi)(i)$ is one-to-one, $i = j$. Thus π is one-to-one.

Let $i \in \{1, ..., m\}$. Since $(\pi \circ \pi^{'})$ is onto, there exists $j \in \{1, ..., m\}$ such that $(\pi \circ \pi^{'})(j) = i$. Thus π is onto. So π is a bijection. Similary, we get ϕ is a bijection.

Conversely, let $\sigma_{t,F} \in Rephyp((m),(n))$ where $t = f(x_{\pi(1)}, ..., x_{\pi(m)})$ where π is a bijection. Then there exists $\pi^{'}$ such that $\pi \circ \pi^{'} = (1) = \pi^{'} \circ \pi$. Let $u = f(x_{\pi'(1)}, ..., x_{\pi'(m)})$, then

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(f) &= \widehat{\sigma}_{t,F}[f(x_{\pi'(1)}, ..., x_{\pi'(m)})] \\ &= f(x_{\pi'(\pi(1))}, ..., x_{\pi'(\pi(m))}) \\ &= f(x_1, ..., x_m) \\ &= \sigma_{id}\end{aligned}$$

and

$$\begin{aligned}(\sigma_{u,H} \circ_r \sigma_{t,F})(f) &= \widehat{\sigma}_{u,H}[f(x_{\pi(1)}, ..., x_{\pi(m)})] \\ &= f(x_{\pi(\pi'(1))}, ..., x_{\pi(\pi'(m))}) \\ &= f(x_1, ..., x_m) \\ &= \sigma_{id}.\end{aligned}$$

Similary, we get $(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) = \sigma_{id} = (\sigma_{u,H} \circ_r \sigma_{t,F})(\gamma)$. □

Lemma 4. *Let $t = x_i \in X_m$ and $F = \gamma(s_1, ..., s_n) \in \gamma\mathcal{F}_{((m),(n))}(X_n)$ where $var(F) \cap X_n = \{x_{b_1}, ..., x_{b_l}\}$ and $j-most(s_{b'_k}) = x_{b_k}$; $j \in \{1, ..., m\}$ and for all $k = 1, ..., l$. Then $\sigma_{t,F} \circ_r \sigma_{u,H} \circ_r \sigma_{t,F} = \sigma_{t,F}$ if and only if it satisfies one of the following conditions:*

(i) *$i = j$; $u = f(u_1, ..., u_m)$ where $u_j = x_j$ and $H = \gamma(h_1, ..., h_n)$ with $h_{b_k} = x_{b'_k}$ for all $k = 1, ..., l$,*

(ii) *$i \neq j$; $u = x_j$ and $H = \gamma(h_1, ..., h_n)$ with $h_{b_k} = x_{b'_k}$ for all $k = 1, ..., l$.*

Proof. Assume that (i), (ii) are not true, that is $u_j \neq x_j$ and $h_{b_k} \neq x_{b'_k}$ all $k = 1, ..., l$. We will show that $(\sigma_{t,F} \circ_r \sigma_{u,H} \circ_r \sigma_{t,F})(\gamma) \neq \sigma_{t,F}(\gamma)$.

Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H} \circ_r \sigma_{t,F})(\gamma) = \widehat{\sigma}_{t,F}[\widehat{\sigma}_{u,H}[F]]$$
$$= \widehat{\sigma}_{t,F}[\gamma(a_1, ..., a_n)]$$

where

$$a_i = R^n_n(h_i, \widehat{\sigma}_{u,H}[s_1], ..., \widehat{\sigma}_{u,H}[s_n]) \text{ all } i = 1, ..., n$$
$$= R^n_n(\gamma(s_1, ..., s_n), \widehat{\sigma}_{x_i,F}[a_1], ..., \widehat{\sigma}_{x_i,F}[a_n])$$
$$= \gamma(S^n_n(s_1, i-most(a_1), ..., i-most(a_n)), ...,$$
$$S^n_n(s_n, i-most(a_1), ..., i-most(a_n))).$$

Since $j - most(s_{b'_k}) = x_{b_k}$ for all $k = 1, ..., l$, we have

$$S^n_n(i-most(s_{b'_k}), \widehat{\sigma}_{x_i,F}[a_1], ..., \widehat{\sigma}_{x_i,F}[a_n])$$
$$= S^n_n(x_{b_k}, i-most(a_1), ..., i-most(a_n))$$
$$= i-most(a_{b_k})$$
$$= i-most(S^n_n(h_{b_k}, \widehat{\sigma}_{u,H}[s_1], ..., \widehat{\sigma}_{u,H}[s_n]))$$
$$= S^n_n(i-most(h_{b_k}), i-most(\widehat{\sigma}_{u,H}[s_1]),$$
$$..., i-most(\widehat{\sigma}_{u,H}[s_n])).$$

Since $u_j \neq x_j$, by Lemma 1, we have

$$i-most(\widehat{\sigma}_{u,H}[s_j]) \neq i-most(s_{b'_k}) = x_{b_k}$$

and

$$h_{b_k} \neq x_{b'_k},$$

so $S^n_n(i-most(h_{b_k}), i-most(\widehat{\sigma}_{u,H}[s_1]), ..., i-most(\widehat{\sigma}_{u,H}[s_n])) \neq x_{b_k}$.

Thus $(\sigma_{t,F} \circ_r \sigma_{u,H} \circ_r \sigma_{t,F})(\gamma) \neq \sigma_{t,F}(\gamma)$.

Conversely, let $\sigma_{u,H} \in Relhyp((m), (n))$ such that $u = f(u_1, ..., u_m)$ where $u_j = x_j$ and $H = \gamma(h_1, ..., h_n)$ with $h_{b_k} = x_{b'_k}$ for all $k = 1, ..., l$.

Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H} \circ_r \sigma_{t,F})(f) = \widehat{\sigma}_{t,F}[\widehat{\sigma}_{u,H}[t]] = \widehat{\sigma}_{t,F}[\widehat{\sigma}_{u,H}[x_i]] = x_i = \widehat{\sigma}_{t,F}(f)$$

and

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H} \circ_r \sigma_{t,F})(\gamma) &= \widehat{\sigma}_{t,F}[\widehat{\sigma}_{u,H}[\gamma(s_1, ..., s_n)]] \\ &= \widehat{\sigma}_{t,F}[R^n_n(\gamma(h_1, ..., h_n), \widehat{\sigma}_{u,H}[s_1], ..., \widehat{\sigma}_{u,H}[s_n])] \\ &= \widehat{\sigma}_{t,F}[\gamma(a_1, ..., a_n)]\end{aligned}$$

where $a_p = S^n_n(h_p, \widehat{\sigma}_{u,H}[s_1], ..., \widehat{\sigma}_{u,H}[s_n])$ for all $p \in \{1, ..., n\}$. Since $j-most(s_{b'_k}) = x_{b_k}$ and $h_{b_k} = x_{b'_k}$ for all $k \in \{1, ..., l\}$, we have

$$\begin{aligned}a_{b_k} &= S^n_n(h_{b_k}, \widehat{\sigma}_{u,H}[s_1], ..., \widehat{\sigma}_{u,H}[s_n]) \\ &= S^n_n(x_{b'_k}, \widehat{\sigma}_{u,H}[s_1], ..., x_{b_k}, ..., \widehat{\sigma}_{u,H}[s_n]) \ ; \text{ where } x_{b_k} \in var(s_{b'_k}). \\ &= x_{b_k}.\end{aligned}$$

Then

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H} \circ_r \sigma_{t,F})(\gamma) &= \widehat{\sigma}_{t,F}[\gamma(a_1, ..., a_n)] \\ &= R^n_n(F, \widehat{\sigma}_{t,F}[a_1], ..., x_{b_k}, ..., \widehat{\sigma}_{t,F}[a_n]) \\ &\quad \text{where } x_{b_k} \in var(s_{b'_k}) \text{ for all } k = 1, ..., l \\ &= \sigma_{t,F}(\gamma).\end{aligned}$$

Therefore $\sigma_{t,F}$ is unit-regular. Similarly, (ii) can be proved as in (i). □

Lemma 5. *Let $t = f(t_1, ..., t_m) \in W_{(m)}(X_m)$ and $F = \gamma(s_1, ..., s_n) \in \gamma\mathcal{F}_{((m),(n))}(X_n)$ such that $var(t) \cap X_m = \{x_{a_1}, ..., x_{a_k}\}$ with $t_{a'_i} = x_{a_i}$ for some $a'_i \in \{1, ..., m\}$; $i \in \{1, ..., k\}$ and $var(F) \cap X_n = \{x_{b_1}, ..., x_{b_l}\}$ with $s_{b'_j} = x_{b_j}$ for some $b'_j \in \{1, ..., n\}$; $j \in \{1, ..., l\}$.*

Then $\sigma_{t,F} \circ_r \sigma_{u,H} \circ_r \sigma_{t,F} = \sigma_{t,F}$ if and only if $u = f(u_1, ..., u_m)$ which $u_{a_1} = x_{a'_1}, ..., u_{a_k} = x_{a'_k}$ and $H = \gamma(h_1, ..., h_n)$ which $h_{b_1} = x_{b'_1}, ..., h_{b_l} = x_{b'_l}$.

Proof. Assume that $\sigma_{t,F} \circ_r \sigma_{u,H} \circ_r \sigma_{t,F} = \sigma_{t,F}$.

Let $u = f(u_1, ..., u_m)$. Suppose that $u_{a_i} \in W_{(m)}(X_m)\backslash\{x_{a'_i}\}$ for some $i \in \{1, ..., k\}$. Let $H = \gamma(h_1, ..., h_n)$. Suppose that $h_{b_j} \in W_{((m),(n))}(X_n)\backslash\{x_{b'_j}\}$ for some $j \in \{1, ..., l\}$.

Consider

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H} \circ_r \sigma_{t,F})(f) &= \widehat{\sigma}_{t,F}[\widehat{\sigma}_{u,H}[t]] \\ &= \widehat{\sigma}_{t,F}[S^m_m(f(u_1, ..., u_m), \widehat{\sigma}_{u,H}[t_1], ..., \widehat{\sigma}_{u,H}[t_m])] \\ &= \widehat{\sigma}_{t,F}[f(w_1, ..., w_m)]\end{aligned}$$

where

$$\begin{aligned}w_i &= S^m_m(u_i, \widehat{\sigma}_{u,H}[t_1], ..., \widehat{\sigma}_{u,H}[t_m]) \\ &= f(v_1, ..., v_m) \;\text{ where }\; v_i = S^m_m(t_i, \widehat{\sigma}_{t,F}[w_1], ..., \widehat{\sigma}_{t,F}[w_m]).\end{aligned}$$

Since $t_{a'_i} = x_{a_i}; i \in \{1, ..., k\}$, we have

$$v_{a'_i} = S^m_m(t_{a'_i}, \widehat{\sigma}_{t,F}[w_1], ..., \widehat{\sigma}_{t,F}[w_m]) = \widehat{\sigma}_{t,F}[w_{a_i}].$$

Since $w_{a_i} = S^m_m(u_{a_i}, \widehat{\sigma}_{u,H}[t_1], ..., \widehat{\sigma}_{u,H}[t_m])$ and $u_{a_i} \neq x_{a'_i}$, we have $w_{a_i} \neq \widehat{\sigma}_{u,H}[t_{a'_i}] = x_{a_1}$. We get $v_{a'_i} = \widehat{\sigma}_{t,F}[w_{a_i}] \neq x_{a_1}$ and then $f(v_1, ..., v_m) \neq t$, this is a contradiction.

Hence $u_{a_i} = x_{a'_i}$ for all $i \in \{1, ..., k\}$.

Similarly, we have

$$(\sigma_{t,F} \circ_r \sigma_{u,H} \circ_r \sigma_{t,F})(\gamma) = \sigma_{t,F}(\gamma).$$

Therefore $h_{b_j} = x_{b'_j}$ for all $j \in \{1, ..., l\}$.

Conversely, let $u = f(u_1, ..., u_m)$ where $u_{a_i} = x_{a'_i}$ for all $i \in \{1, ..., k\}$ and $h = \gamma(h_1, ..., h_n)$ where $h_{b_j} = x_{b'_j}$ for all $j \in \{1, ..., l\}$. Then $(\sigma_{t,F} \circ_r \sigma_{u,H} \circ_r \sigma_{t,F})(f)$

$= \widehat{\sigma}_{t,F}[f(w_1,...,w_m)]$ where $w_i = S^m_m(u_i, \widehat{\sigma}_{u,H}[t_1],...,\widehat{\sigma}_{u,H}[t_m])$. Since $u_{a_i} = x_{a'_i}$, so

$$\begin{aligned} w_{a_i} &= S^m_m(u_{a_i}, \widehat{\sigma}_{u,H}[t_1],...,\widehat{\sigma}_{u,H}[t_m]) \\ &= S^m_m(x_{a'_i}, \widehat{\sigma}_{u,H}[t_1],...,\widehat{\sigma}_{u,H}[t_m]) \\ &= \widehat{\sigma}_{u,H}[t_{a'_i}] \\ &= x_{a_i}. \end{aligned}$$

We get

$$\widehat{\sigma}_{t,F}[f(w_1,...,w_m)] = S^m_m(f(t_1,..,t_m), \widehat{\sigma}_{t,F}[w_1],...,\widehat{\sigma}_{t,F}[w_m]) = f(t_1,...,t_m).$$

Hence $(\sigma_{t,F} \circ_r \sigma_{u,H} \circ_r \sigma_{t,F})(f) = (\sigma_{t,F})(f)$.

Similarly, we get $(\sigma_{t,F} \circ_r \sigma_{u,H} \circ_r \sigma_{t,F})(\gamma) = (\sigma_{t,F})(\gamma)$. □

Let $\sigma_{t,F} \in Relhyp((m),(n))$, we denote

$R'_X := \{\sigma_{t,F} \mid t = x_i \in X_m$ and $F = \gamma(s_1,...,s_n)$ with $var(F) = \{x_{b_1},...,x_{b_l}\}$
such that $i{-}most(s_{b'_k}) = x_{b_k}$ for all $k = 1,...,l$ and some distinct $b'_1,...,b'_l \in \{1,...,n\}$ where $i \in \{1,...,m\}\}$.

Theorem 2. *$R'_X \cup R_T$ is the set of all unit-regular elements in $Relhyp((m),(n))$.*

Proof. Let $\sigma_{t,F} \in R'_X \cup R_T$.

Case I. $\sigma_{t,F} \in R'_X$. Then $t = x_i$ and $F = \gamma(s_1,...,s_n) \in \gamma\mathcal{F}_{((m),(n))}(X_n)$ with $var(F) = \{x_{b_1},...,x_{b_l}\}$ such that $i-most(s_{b'_k}) = x_{b_k}$ for all $k = 1,...,l$ and some distinct $b'_1,...,b'_l \in \{1,...,n\}$ where $i \in \{1,...,m\}$.

Choose $\sigma_{u,H} \in U(Relhyp((m),(n)))$ such that

$$u = f(u_1,...,u_m) = f(x_{\mu(1)},...,x_{\mu(m)})$$

and $H = \gamma(h_1,...,h_n) = \gamma(x_{\nu(1)},...,x_{\nu(n)})$ for some $\mu \in S_m, \nu \in S_n$ such that $\mu(i) = i$ and $\nu(b'_1) = b_1,...,\nu(b'_n) = b_n$.

Then $u_i = x_{\mu(i)} = x_i$ and $H_{b'_j} = x_{\nu(b'_j)} = x_{b_j}$ for all $j \in \{1, .., l\}$.

By Lemma 4, we get

$$\sigma_{t,F} \circ_r \sigma_{u,H} \circ_r \sigma_{t,F} = \sigma_{t,F}.$$

Case II. $\sigma_{t,F} \in R_T$ where $t = f(t_1, \ldots, t_m)$ and $F = \gamma(s_1, ..., s_n)$ with $var(t)$ $= \{x_{a_1}, ..., x_{a_k}\}$ and $var(F) = \{x_{b_1}, ..., x_{b_l}\}$ such that $t_{a'_i} = x_{a_i}$ and $s_{b'_j} =$ x_{b_j} for all $i = 1, ..., k, j = 1, ..., l$ for some distinct $a'_1, ..., a'_k \in \{1, ..., m\}$ and for some distinct $b'_1, ..., b'_l \in \{1, ..., n\}$.

Choose $\sigma_{v,G} \in U(Relhyp((m), (n)))$ such that

$$v = f(v_1, ..., v_m) = f(x_{\beta(1)}, \cdots, x_{\beta(m)})$$

and

$$G = \gamma(g_1, ..., g_n) = \gamma(x_{\alpha(1)}, ..., x_{\alpha(n)})$$

for some $\beta \in S_m, \alpha \in S_n$ such that

$$\beta(a_1) = a'_1, ..., \beta(a_m) = a'_m$$

and

$$\alpha(b'_1) = b_1, ..., \alpha(b'_n) = b_n.$$

Then $v_{a'_i} = x_{\beta(a'_i)} = x_{a_i}$ for all $i \in \{1, .., k\}$ and $g_{b'_j} = x_{\alpha(b'_j)} = x_{b_j}$ for all $j \in \{1, .., l\}$.

By Lemma 5, we get

$$\sigma_{t,F} \circ_r \sigma_{v,G} \circ_r \sigma_{t,F} = \sigma_{t,F}.$$

Hence $\sigma_{t,F}$ is a unit-regular element in $Relhyp((m), (n))$. □

3. Conclusion

In the present paper, we began with the definition of a relational hypersubstitution for algebraic systems of type (τ, τ') and introduced the monoid $(Relhyp(\tau, \tau'), \circ_r, \sigma_{id})$.

Finally, we determined all unit-regular elements on the monoid$(Relhyp((m),(n)), \circ_r, \sigma_{id})$ for arbitrary natural numbers $m, n \geq 2$.

We concluded that $\sigma_{t,F} \in Relhyp((m),(n))$ is unit-regular if and only if $\sigma_{t,F} \in R'_X \cup R_T$ where

$$R'_X := \{\sigma_{t,F} \ |t = x_i \in X_m$$

and

$$F = \gamma(s_1, ..., s_n) \text{ with} var(F) = \{x_{b_1}, ..., x_{b_l}\}$$

such that $i-most(s_{b'_k}) = x_{b_k}$ for all $k = 1, ..., l$ and somedistinct $b_1^{'}, ..., b_l^{'} \in \{1, ..., n\}$ where $i \in \{1, ..., m\}\}$ and $R_T := \{\sigma_{t,F} \mid t = f(t_1, \ldots, t_m)$ and $F = \gamma(s_1, ..., s_n)$ with $var(t) = \{x_{a_1}, ..., x_{a_k}\}$ and $var(F) = \{x_{b_1}, ..., x_{b_l}\}$ such that $t_{a'_i} = x_{a_i}$ and $s_{b'_j} = x_{b_j}$for all $i = 1, ..., k, j = 1, ..., l$ for some distinct $a'_1, ..., a'_k \in \{1, ..., m\}$ and for some distinct $b'_1, ..., b'_l \in \{1, ..., n\}\}$.

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